Quantum Period Theorem

Note Title 11/18/2019

$X = \mathbb{P}^2$, $\check{X} = (\mathbb{C}^*)^2$ w/ $W_0 = z_1 + z_2 + \frac{1}{z_1 z_2}$

$\Xi \in H^0(\check{X}, \mathrm{Re}(W/\hbar) \ll 0, \mathbb{Z})$

$\rightsquigarrow f = \int_\Xi e^{W/\hbar}\,\Omega$ satisfies the ODE $\left(\hbar \frac{\partial}{\partial \hbar}\right)^3 f = 27\hbar^{-3} f$

converges

Choose Ξ_0, Ξ_1, Ξ_2 s.t the corresponding f_0, f_1, f_2

$\Xi_0 \cong T^2 = \{|z_1| = |z_2| = \delta\} \subseteq (\mathbb{C}^*)^2$

$f_1 \sim \log\hbar$, $f_2 \sim (\log\hbar)^2$

Explicit coefficients of Ξ_i are determined by $J^{small}_{\mathbb{P}^2}$.

Recall that

$W_k(\Theta) = \sum_{\mathfrak{h}:\, MI(\mathfrak{h}) = 2} \mathrm{Mono}(\mathfrak{h})$, $\mathrm{Mono}(h) = \mathrm{Mult}(\mathfrak{h})\, z^{\Delta(h)}\, u_{I(h)}$

$I = \{t_{i_1}, \dots, t_{i_n}\} \subseteq \{1, \dots, k\}$

$t_I = t_{i_1} \cdots t_{i_n}$

$\frac{1}{(2\pi i)^2} \int_{\Xi_0} e^{W_k/\hbar} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$ well-defined

- For another choice of $\Theta, P_1, \dots, P_n$, $W_k(\Theta) \longmapsto \theta W_k(\Theta)$, $\theta \in V_{\Sigma, k}$

$e^{W_k/\hbar}\Omega - e^{\theta W_k/\hbar}\Omega$ is d-exact

$$= \sum_{m=0}^{\infty} \mathrm{Const}(W_k^m) \frac{\hbar^{-m}}{m!}$$

$\mathrm{Const}(W_k^m) \ni \prod_i \mathrm{Mono}(h_i)^{m_i}$, $\sum_i m_i = m$

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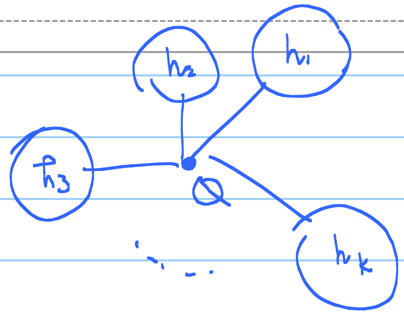
Const term $\Leftrightarrow$ balancing condition at Q

$\hbar =$



balancing conditions at other vertices of $\hbar_i$ are automatic

$\Leftrightarrow$ balancing condition of $\hbar$

i.e. $\gamma\left(\sum_i m_i|\Delta(\hbar_i)|\right) = 0$.

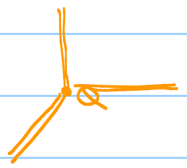
$\Uparrow$ $\dim_i = |\Delta| - n - \nu - 2$

$\mathcal{M}^{trop}_{\Delta,n}(X_\Sigma, P_1, \cdots, P_k, \psi^{m-2} Q)$

$n_i(x) = \#$ of unbounded edges in direction m_i

$$\mathrm{Mult}(\hbar) := \frac{1}{n_0(x)!\, n_1(x)!\, n_2(x)!} \prod_i \mathrm{Mono}(\hbar_i)^{m_i}$$

ex. $\hbar =$



$\in \mathcal{M}^{trop}_{(2,2,2),0}(\mathbb{P}^2, P_1, \cdots, P_k, \psi^4 Q)$

$$\mathrm{Mult}(\hbar) = \left(\frac{1}{2!}\right)^3 = \frac{1}{8}$$

Conversely, every $\hbar \in \mathcal{M}^{trop}_{\Delta,n}(X_\Sigma, P_1, \cdots, P_k, \psi^{\nu} Q)$. $n = |\Delta| - \nu - 2$ s.t. $\dim \mathcal{M} = 0$

is gluing of Maslov index 2 tropical discs stop at Q.

pf: $MI(\hbar_i) \leq 2$ otherwise $\hbar$ is NOT rigid.

Let $n_i = \#$ of marked points on $\hbar_i$

$$\sum_{i=1}^{\nu+2} \frac{MI(h_i)}{2} = \sum_i \left(|\Delta(\hbar_i)| - n_i\right) = |\Delta(h)| - n = \nu + 2$$

i.e. $MI(\hbar_i) = 2$, $\forall i$

Therefore, $\frac{1}{(2\pi i)^2}\int_{T^2} e^{W_k/\hbar}\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}$

$$= 1+\sum_{m\geq 1}\mathrm{Const}\left(W_k^m\right)\frac{\hbar^{-m}}{m!}$$

$$= 1+\sum_{m\geq 1}\Big(\underbrace{\prod_i \mathrm{Mono}(h_i)^{m_i}}_{=0}\Big)\boxed{C^m_{m_1,\cdots,m_n}\frac{\hbar^{-m}}{m!}} = \frac{\hbar^{-m}}{m_1!\cdots m_n!},$$

if $m_i > 1$, since $t_i^2 = 0$

unless h_i has one unbounded edge

$$= 1+\sum_{m\geq 1}\underbrace{\frac{(zP_0)^{n_0(x)}(zP_1)^{n_1(x)}(zP_2)^{n_2(x)}}{n_0(x)!\, n_1(x)!\, n_2(x)!}\Big(\prod_{h_i:\text{ not basic}}\mathrm{Mono}(h_i)\Big)}_{=\langle P_{i_1},\cdots,P_{i_n},\psi^\nu Q\rangle_d^{trop} t_I K^d}\hbar^{-m}$$

$n = |\Delta|-\nu-2$, $m=\nu+2\geq 2$

From algebraic geometry, $\langle P_{i_1},\cdots,P_{i_n},\psi^\nu Q\rangle_d^{trop} = \langle P_{j_1},\cdots,P_{j_n},\psi^\nu Q\rangle_d^{trop}$

$$= 1+\sum_{d\geq 1}\sum_{\nu\geq 0}\langle P_1,\cdots,P_{|\Delta|-\nu-2},\psi^\nu Q\rangle_d^{trop}\Big(\sum_{I:\,|I|=|\Delta|-2-\nu} t_I\Big)K^d\hbar^{-(\nu+2)}$$

$$= 1+\sum_{d\geq 1}\sum_{\nu\geq 0}\langle P_1,\cdots,P_{|\Delta|-\nu-2},\psi^\nu Q\rangle_d^{trop}\hbar^{-(\nu+2)}K^d\frac{y_2^{|\Delta|-\nu-2}}{(|\Delta|-\nu-2)!},\quad y_2=\sum_{i=1}^k t_i$$

Question: How do we understand $\int_{\Xi_i} e^{W_k/\hbar}\Omega$?